Kiel University
Christian-Albrechts-Universität zu Kiel

Scattering and Diffraction of Complex-Source Beams by Canonical Objects

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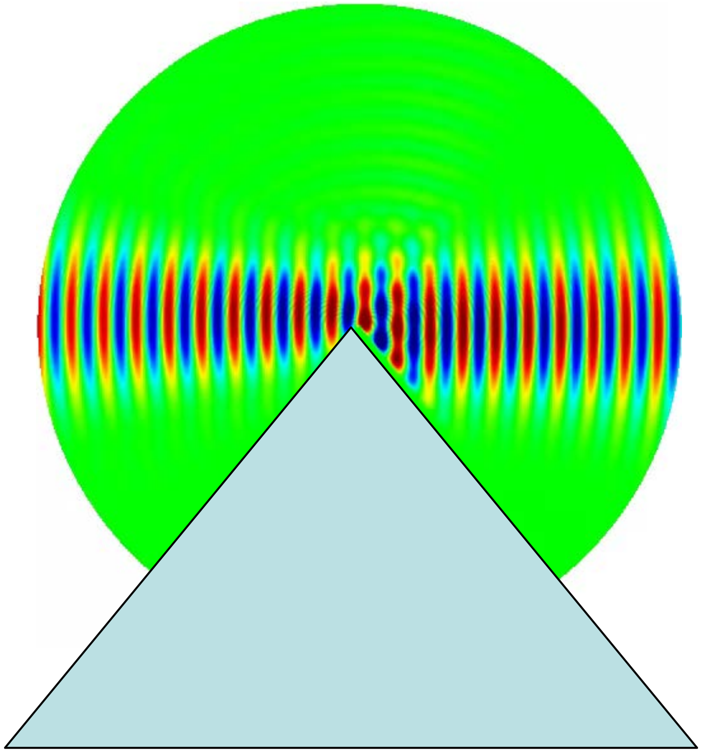
Kiel: ~ 250,000 residents, Capital of Schleswig-Holstein

Christian-Albrechts-Universität zu Kiel (about 25,000 students), founded 1665 by Christian Albrecht, Duke of Gottorf-Holstein

Outline

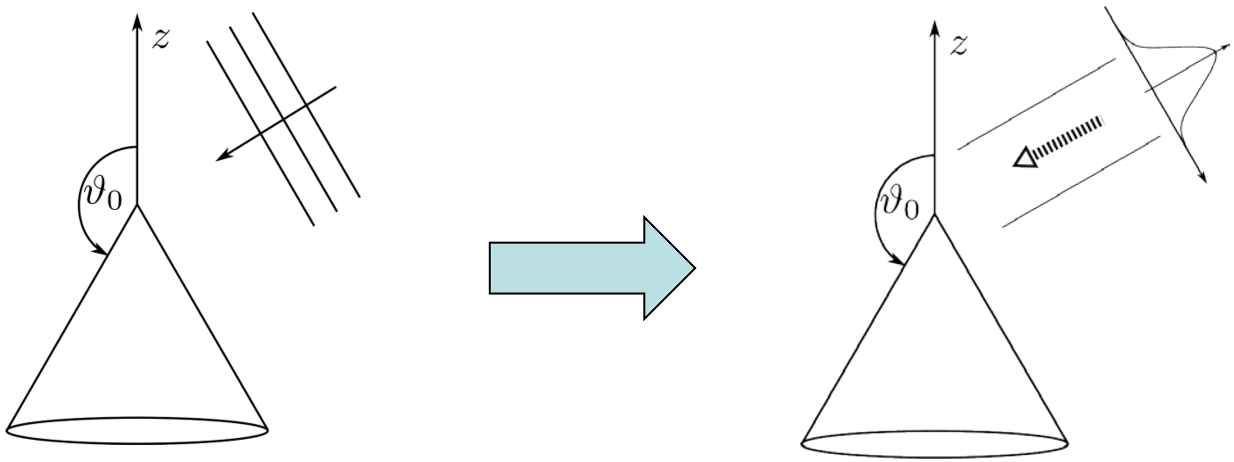
- Problem and Motivation
- Spherical-Multipole Analysis
- Uniform Complex-Source Beam (CSB)
- Uniform CSB Diffraction by a Cone
- Numerical Example: CSB-Scattering by a Sector
- Conclusions



Understanding the scattering by canonical objects including structures with geometrical singularities (tips, edges, …) is important for several applications, e.g, improving asymptotic methods (GTD, UTD).

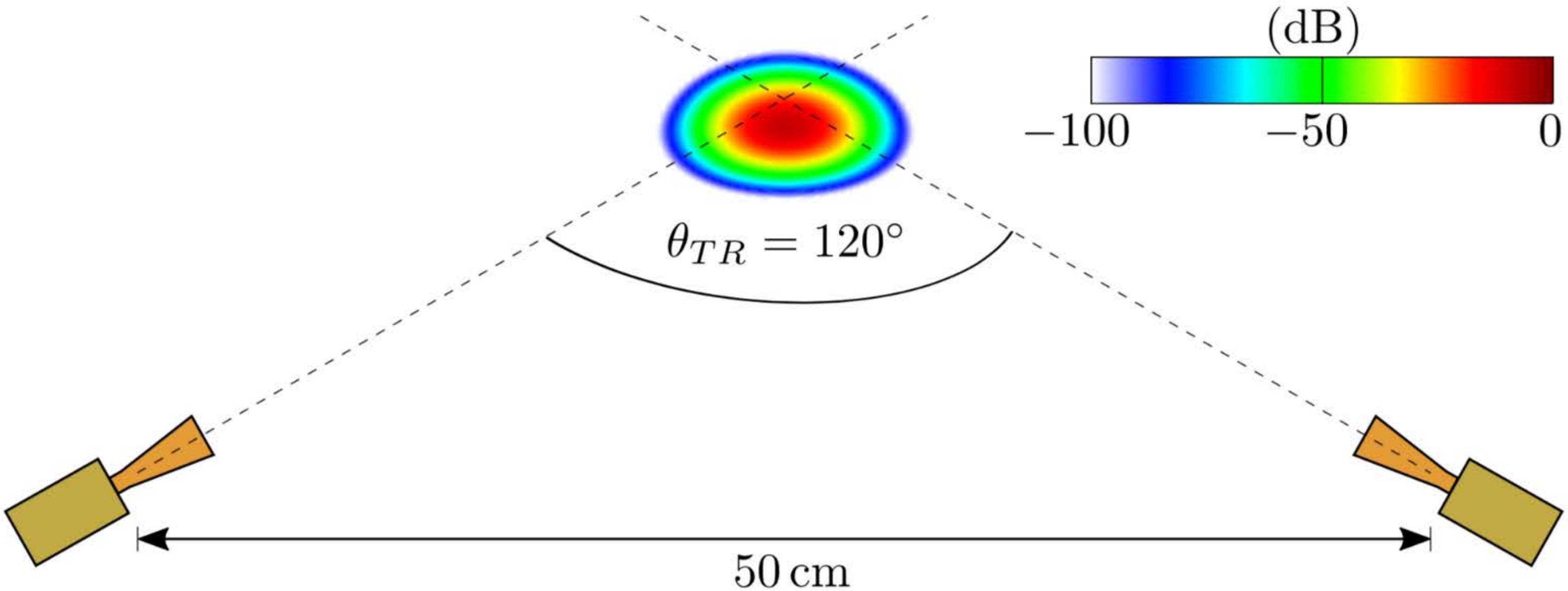
- Multipole analysis of full plane-wave scattering by semi-infinite structures shows convergence problems.

- Solution:
  Use a localized plane wave as the incident field!



- Realization by a (uniform) complex-source beam (CSB).

- Complex-source beams can be applied to enhance the analysis in the context of small-particles measurement using microwaves.



(A. Reinhardt et al., IEEE Trans MTT, 65 (2017) 5244 – 5250)

Spherical-Multipole Analysis

4 Einleitung. §. 1.

Führt man in T sogenannte Kugelcoordinaten ein, d. h. setzt

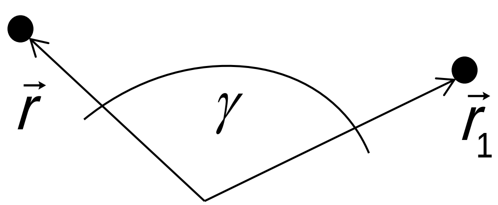
$$x = r\cos\theta \qquad x_1 = r_1\cos\theta_1$$
$$y = r\sin\theta\cos\psi \qquad y_1 = r_1\sin\theta_1\cos\psi_1$$
$$z = r\sin\theta\sin\psi \qquad z_1 = r_1\sin\theta_1\sin\psi_1,$$

wo r und r_1 positiv, θ und θ_1 zwischen 0 und π, ψ und ψ_1 zwischen 0 und 2π genommen werden, so verwandelt es sich in

$$T = \sum_{n=0}^{n=\infty} \frac{r_1^n}{r^{n+1}} P^{(n)}(\cos\gamma)$$

verbunden, und man erhält

$$T = \frac{1}{\sqrt{r^2 - 2rr_1\cos\gamma + r_1^2}}.$$



Die Entwickelung, von der oben die Rede war, die sich bei Laplace*) und Legendre**) findet, besteht darin, dass man T nach aufsteigenden Potenzen der kleineren von den beiden Entfer-ngen r und r_1 — sie sei r_1 — und nach absteigenden der grösse- r ordnet: der Coefficient von $\frac{r_1^n}{r^{n+1}}$, der also nur von $\cos\gamma$ ab-ngt, ist dann die n^{te} Kugelfunction. Wählt man als Functions-chen für dieselbe mit Dirichlet***) den Buchstaben $P^{(n)}$, oder wo eine Verwechselung mit Potenzen unmöglich ist, und fügt sem das Argument $\cos\gamma$ in Parenthese hinzu, so ist P^n durch Gleichung

$$T = \sum_{n=0}^{n=\infty} \frac{r_1^n}{r^{n+1}} P^{(n)}(\cos\gamma)$$

definirt, und wird eine ganze Function n^{ten} Grades von $\cos\gamma$; ihren genauen Werth findet man unten §. 3.

*) Mémoires de Math. et de Phys. Année 1782 no. X p. 188.

**) Savans étrangers Tome X no. 10, p. 419.

***) Crelle, Journal f. Math. Bd. XVII: Sur les séries dont le terme général dépend de deux angles, et qui servent à exprimer des fonctions arbitraires entre des limites données, S. 35.

E. Heine “Handbuch der Kugelfunktionen”
Verlag von Georg Reimer: *Berlin 1861*

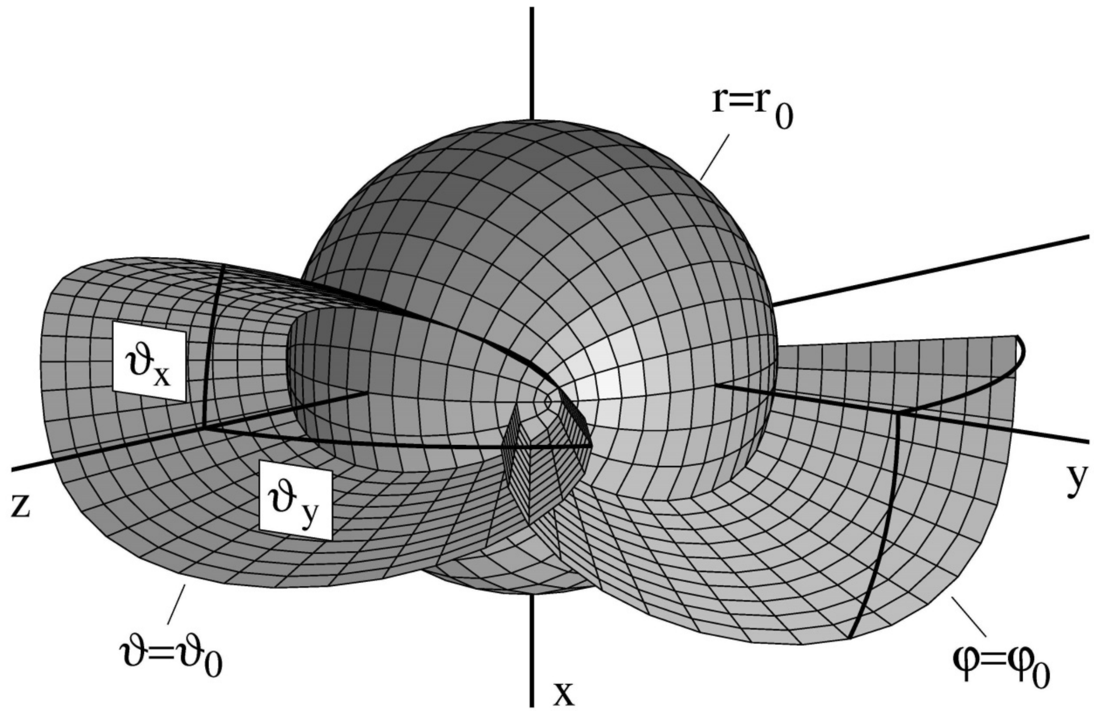
Sphero-Conal Coordinates:

$$x = r \sin\vartheta \cos\varphi$$

$$y = r\sqrt{1-k^2\cos^2\vartheta}\,\sin\varphi$$

$$z = r\cos\vartheta\sqrt{1-k'^2\sin^2\varphi}$$

$$0 \le k, k' \le 1; \quad k^2 + k'^2 = 1$$



$k = 1$: spherical coordinates

$\vartheta_0 = \vartheta_x = \pi$: plane angular sector

$\vartheta_0 = \pi, k^2 = 0.5$: quarter plane

Scaling coefficients:

$$s_\vartheta = \frac{1}{r}\left|\frac{\partial \mathbf{r}}{\partial \vartheta}\right| \quad ; \quad s_\varphi = \frac{1}{r}\left|\frac{\partial \mathbf{r}}{\partial \varphi}\right|$$

Spherical-Multipole Analysis

Spherical Multipole Expansion:

$$\mathbf{E}(\mathbf{r}) = \sum_{\sigma} \alpha_{\sigma} \mathbf{N}_{\sigma}(\mathbf{r}) + \frac{Z}{j} \sum_{\tau} \beta_{\tau} \mathbf{M}_{\tau}(\mathbf{r})$$

$$\mathbf{H}(\mathbf{r}) = \frac{j}{Z} \sum_{\sigma} \alpha_{\sigma} \mathbf{M}_{\sigma}(\mathbf{r}) + \sum_{\tau} \beta_{\tau} \mathbf{N}_{\tau}(\mathbf{r})$$

$$\left(Z = \sqrt{\mu_0 / \varepsilon_0} \right)$$

Spherical Multipole Functions:

$$\mathbf{N}_{\nu} = (\mathbf{r} \times \nabla) \Psi_{\nu}$$

$$\mathbf{M}_{\nu} = \frac{1}{\kappa} \nabla \times \mathbf{N}_{\nu}$$

$$\left(\kappa = \omega \sqrt{\mu_0 \varepsilon_0} \right)$$

Helmholtz equation:

$$\Delta \Psi_{\nu} + \kappa^2 \Psi_{\nu} = 0$$

$$\Psi_{\sigma}(r, \vartheta, \varphi) \Big|_{\vartheta = \vartheta_0} = 0$$

$$\left. \frac{\partial \Psi_{\tau}(r, \vartheta, \varphi)}{\partial \vartheta} \right|_{\vartheta = \vartheta_0} = 0$$

Spherical-Multipole Analysis

$$\Delta\Psi_\nu + \kappa^2\Psi_\nu = 0$$

$$\Psi_\nu(r,\vartheta,\varphi) = z_\nu(\kappa r)\,Y_\nu(\vartheta,\varphi)$$

$$z_\nu(\kappa r) = \sqrt{\frac{2}{\pi\kappa r}}\,Z_{\nu+1/2}(\kappa r)$$ Spherical Bessel functions

$$(\mathbf{r}\times\nabla)^2 Y_\nu(\vartheta,\varphi) + \nu(\nu+1)Y_\nu(\vartheta,\varphi) = 0$$

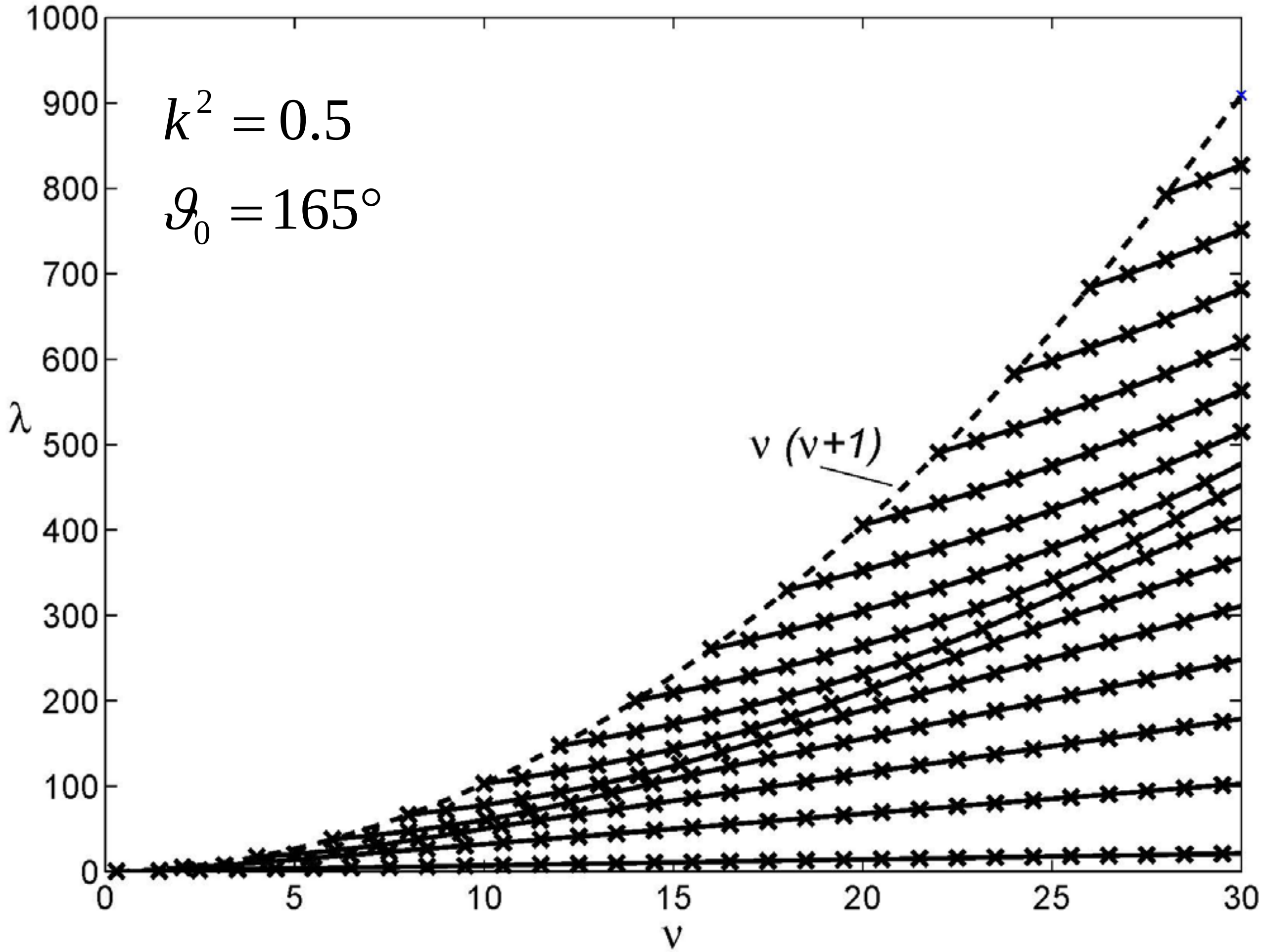
$$Y_\nu(\vartheta,\varphi) = \Theta_\nu(\vartheta)\,\Phi_\nu(\varphi)$$ Lamé products

$$\sqrt{1-k^2\cos^2\vartheta}\,\frac{d}{d\vartheta}\left(\sqrt{1-k^2\cos^2\vartheta}\,\frac{d\Theta_\nu}{d\vartheta}\right) + \left[\nu(\nu+1)\left(1-k^2\cos^2\vartheta\right) - \lambda\right]\Theta_\nu = 0$$

$$\sqrt{1-k'^2\sin^2\varphi}\,\frac{d}{d\varphi}\left(\sqrt{1-k'^2\sin^2\varphi}\,\frac{d\Phi_\nu}{d\varphi}\right) + \left[\lambda - \nu(\nu+1)k'^2\sin^2\varphi\right]\Phi_\nu = 0$$

Spherical-Multipole Analysis

Eigenvalue curves (Dirichlet eigenvalues)

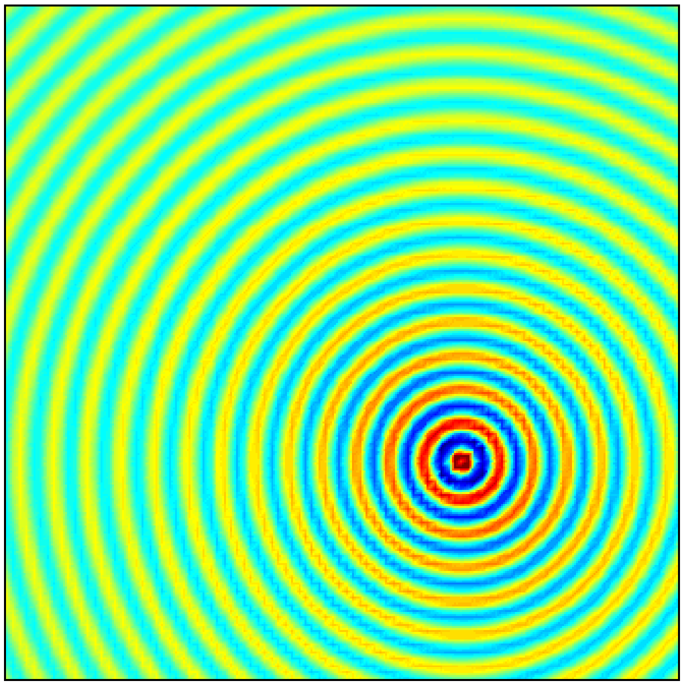


Uniform Complex-Source Beam (Uniform CSB)

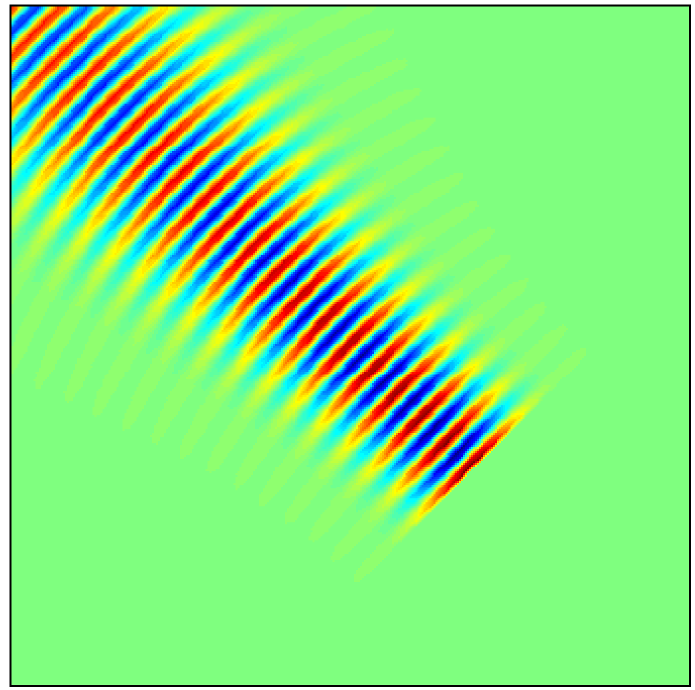
3D free-space scalar Green's function $\mathbf{r} = \{r, \vartheta, \varphi\}$; time-factor $e^{-\mathrm{i}\omega t}$

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

Outwardly travelling spherical waves



$(r', \vartheta', \varphi') = (5\lambda, -45^\circ, 0^\circ)$



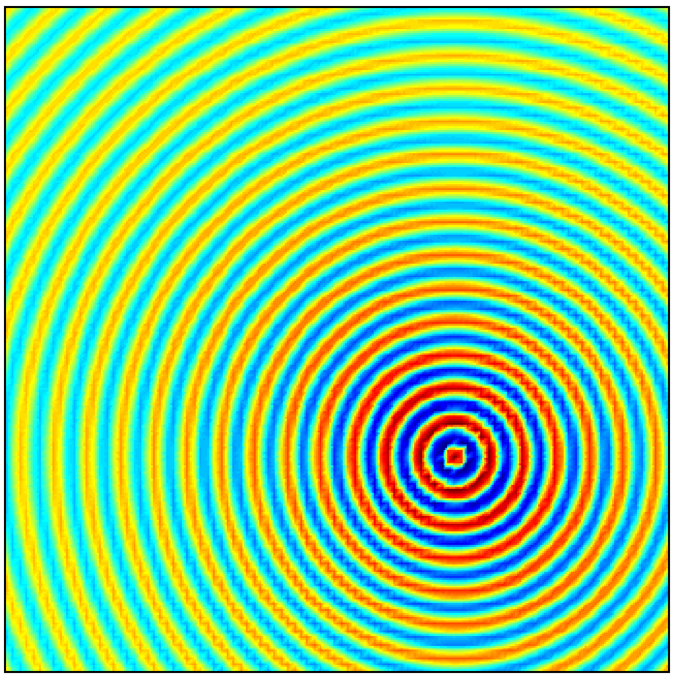
$(r', \vartheta', \varphi') = ((5 - \mathrm{i}10)\lambda, -45^\circ, 0^\circ)$

Uniform Complex-Source Beam (Uniform CSB)

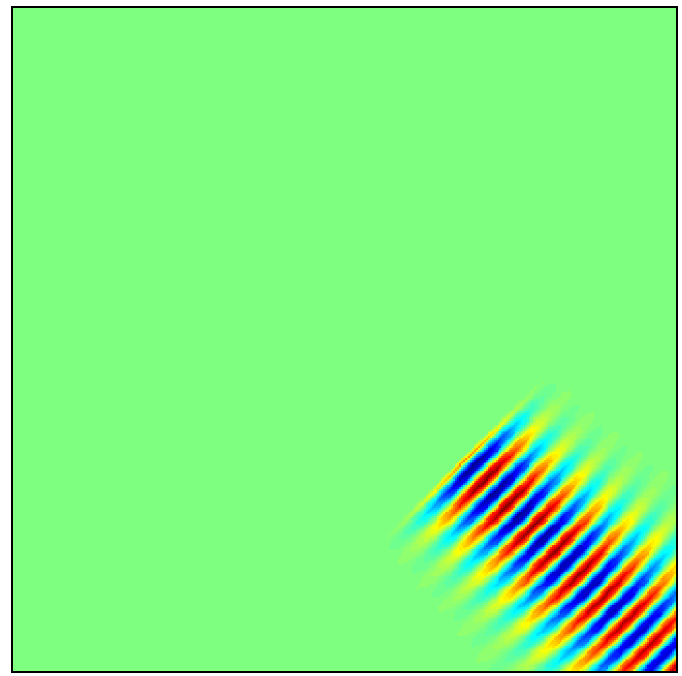
3D free-space Green's function $\mathbf{r} = \{r, \vartheta, \varphi\}$; time-factor $e^{-\mathrm{i}\omega t}$

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \frac{e^{-\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

Inwardly travelling spherical waves



$$(r', \vartheta', \varphi') = (5\lambda, -45^\circ, 0^\circ)$$



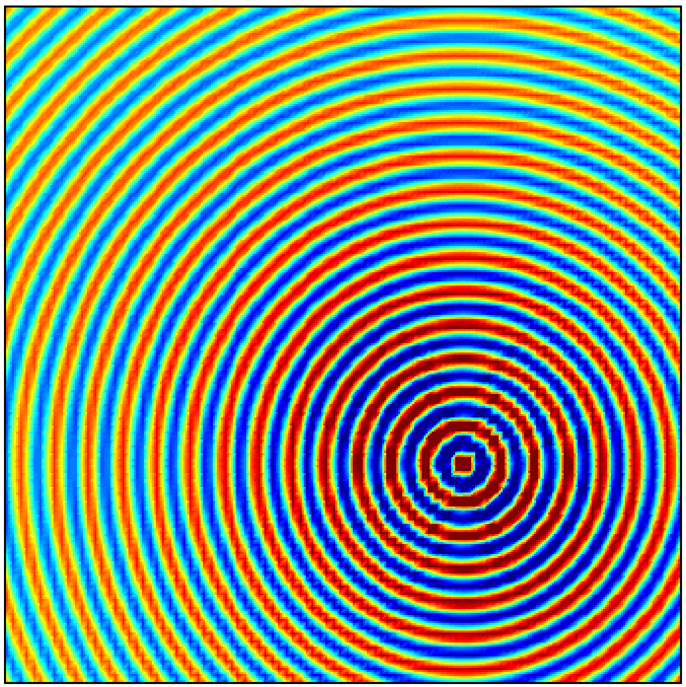
$$(r', \vartheta', \varphi') = ((5 - \mathrm{i}10)\lambda, -45^\circ, 0^\circ)$$

Uniform Complex-Source Beam (Uniform CSB)

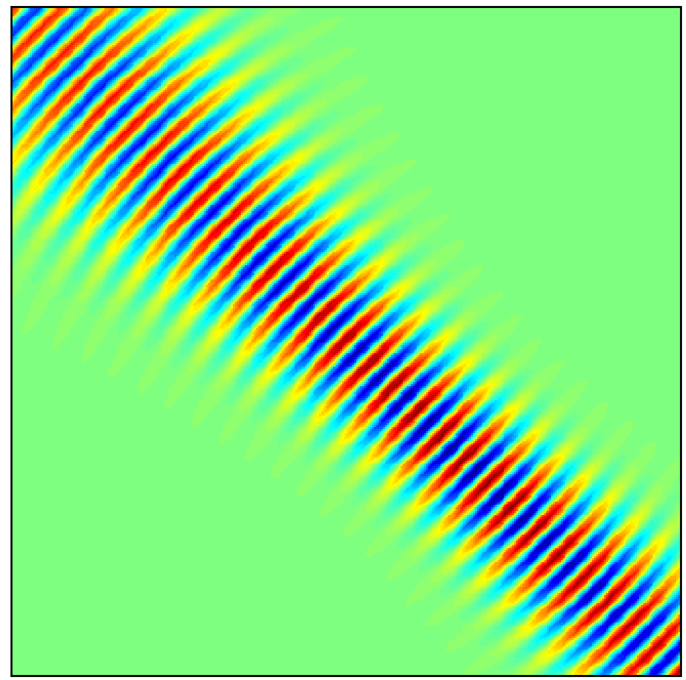
3D free-space Green's function $\mathbf{r} = \{r, \vartheta, \varphi\}$; time-factor $e^{-\mathrm{i}\omega t}$

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\left[\frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} + \frac{e^{-\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right]$$

Uniform representation



$$(r',\vartheta',\varphi') = (5\lambda, -45^\circ, 0^\circ)$$



$$(r',\vartheta',\varphi') = ((5-\mathrm{i}10)\lambda, -45^\circ, 0^\circ)$$

Uniform Complex-Source Beam (CSB)

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{r}' = (\mathrm{i}b)\hat{\mathrm{z}} \quad \text{(Example)}$$

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{(z-\mathrm{i}b)^2 + x^2 + y^2}$$

$$\simeq \pm\left[(z-\mathrm{i}b) + \frac{1}{2}\frac{x^2+y^2}{(z-\mathrm{i}b)}\right] \quad \text{(near } z\text{-axis, 'paraxial')}$$

$$G(\mathbf{r},\mathbf{r}') \simeq \frac{1}{4\pi(z-\mathrm{i}b)} e^{\mathrm{i}k(z-\mathrm{i}b)+\frac{1}{2}\frac{\mathrm{i}k\left(x^2+y^2\right)}{(z-\mathrm{i}b)}}$$

$$\simeq \frac{1}{4\pi(z-\mathrm{i}b)} e^{\mathrm{i}k(z-\mathrm{i}b)+\frac{\mathrm{i}k\left(x^2+y^2\right)}{2\left(z+b^2/z\right)}-\frac{x^2+y^2}{2\,b/k\left(1+z^2/b^2\right)}}$$

Uniform Complex-Source Beam (CSB)

Paraxial Approximation of the CSB solution:

$$G(\mathbf{r},\mathbf{r}') \simeq \frac{e^{kb}}{4\pi(z-\mathrm{i}b)} e^{\mathrm{i}kz+\frac{\mathrm{i}k(x^2+y^2)}{2\underbrace{\left(z+b^2/z\right)}_{K(z)}}-\frac{x^2+y^2}{\underbrace{2b/k\left(1+z^2/b^2\right)}_{[W(z)]^2}}}$$

Exponentially weighted Gaussian beam with:

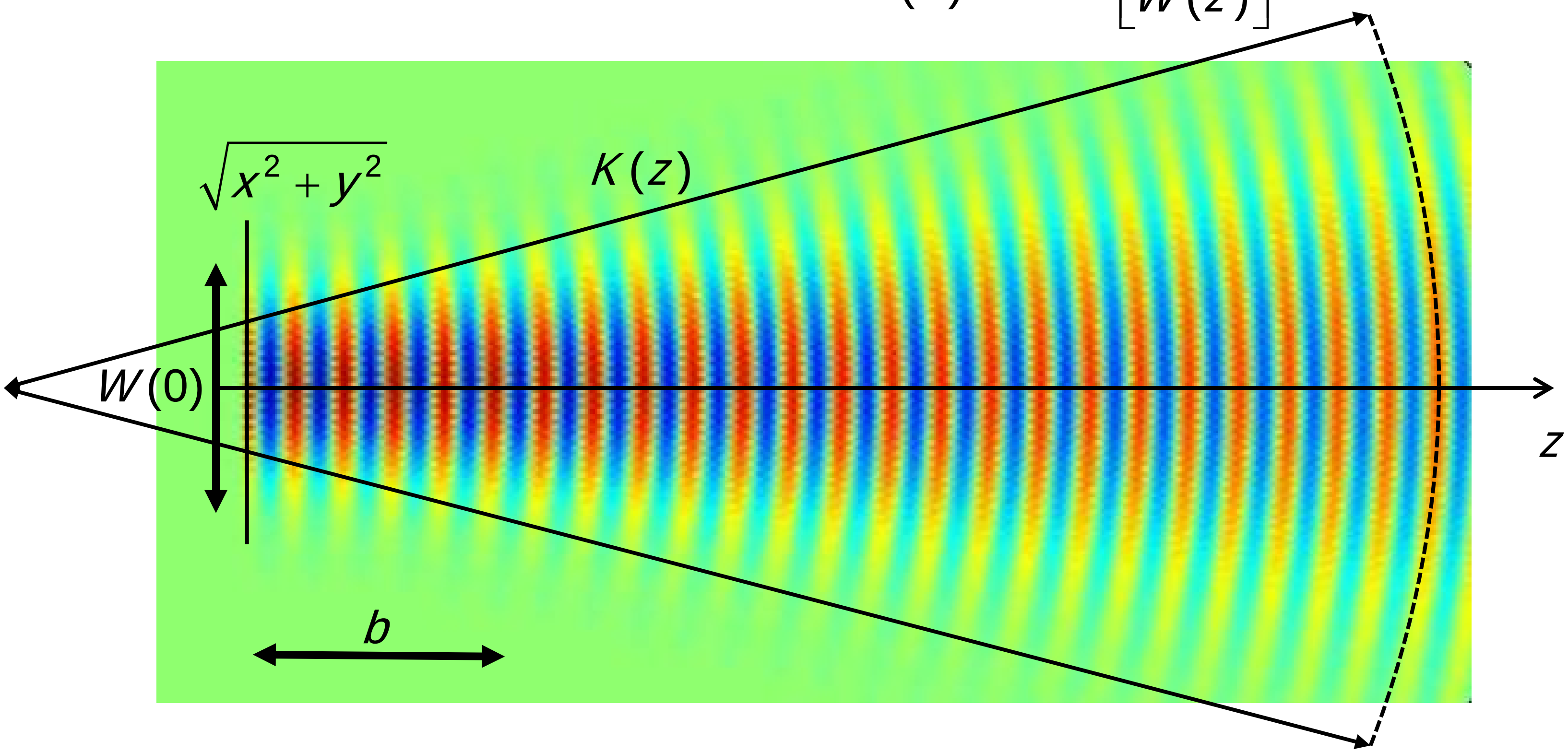
Wave-front radius of curvature: $K(z) = z + b^2/z$

Beam-width: $W(z) = W(0)\sqrt{1+z^2/b^2}$

Beam-width at z=0 (waist): $W(0) = \sqrt{b/k}$

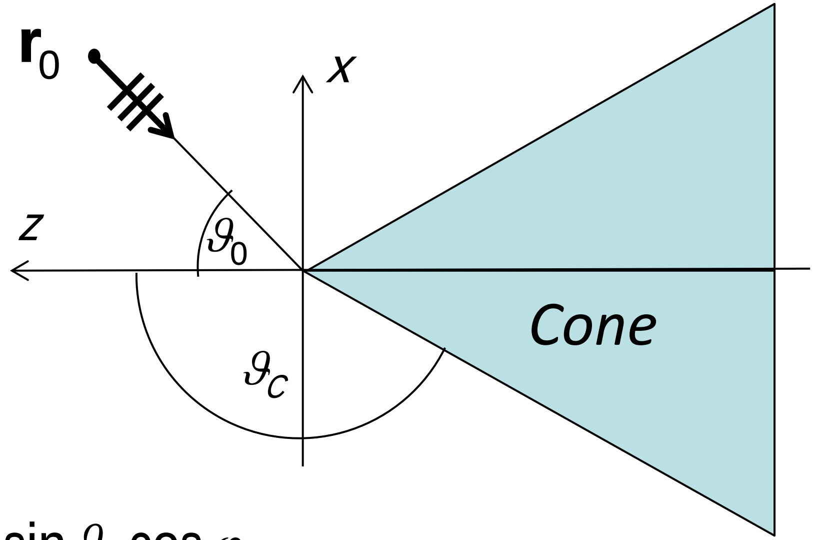
Complex-Source Beam (CSB)

$$G(\mathbf{r},\mathbf{r}') \simeq \frac{e^{kb}}{4\pi(z-\mathrm{i}b)}\, e^{\mathrm{i}kz+\frac{\mathrm{i}k\left(x^2+y^2\right)}{2\underbrace{\left(z+b^2/z\right)}_{K(z)}}-\frac{x^2+y^2}{\underbrace{2b/k\left(1+z^2/b^2\right)}_{[W(z)]^2}}}$$



Uniform CSB Diffraction by a Cone

CSB travelling towards the tip:



$$\mathbf{r}' = \mathbf{r}_0 + \mathrm{i}\mathbf{b}$$

$$\mathbf{r}_0 = \left(r_0, \vartheta_0, \varphi_0\right)$$

$$\mathbf{b} = \left(b, \vartheta_b = \pi - \vartheta_0, \varphi_b = \varphi_0 + \pi\right)$$

$$x' = r_0 \sin\vartheta_0 \cos\varphi_0 + \mathrm{i}b \sin\vartheta_b \cos\varphi_b = \left(r_0 \text{-} \mathrm{i}b\right) \sin\vartheta_0 \cos\varphi_0$$

$$y' = r_0 \sin\vartheta_0 \sin\varphi_0 + \mathrm{i}b \sin\vartheta_b \sin\varphi_b = \left(r_0 \text{-} \mathrm{i}b\right) \sin\vartheta_0 \sin\varphi_0$$

$$z' = r \cos\vartheta_0 + \mathrm{i}b \cos\vartheta_b = \left(r_0 \text{-} \mathrm{i}b\right) \cos\vartheta_0$$

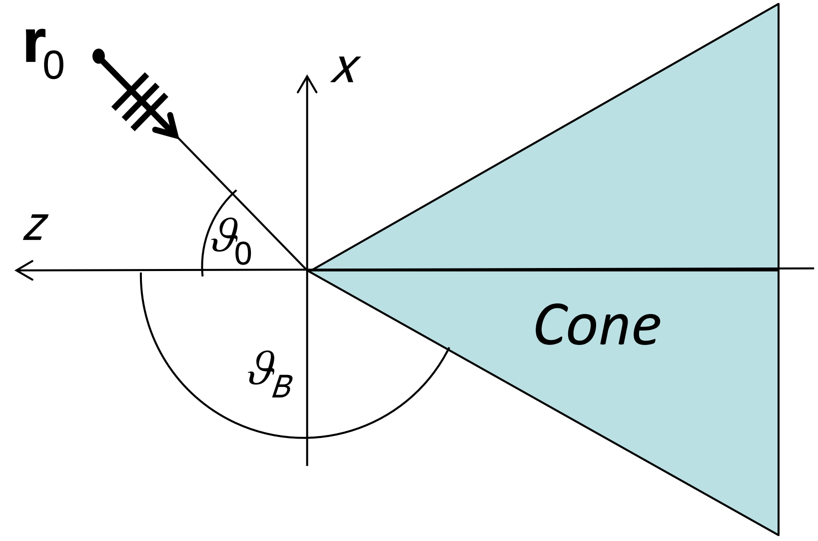
$$r' = r_0 - \mathrm{i}b$$

$$\vartheta' = \vartheta_0$$

$$\varphi' = \varphi_0$$

Uniform CSB Diffraction by a Cone

Green's function:



$$\Delta G_{\substack{s\\h}}\left(\mathbf{r},\mathbf{r}'\right)+k^2 G_{\substack{s\\h}}\left(\mathbf{r},\mathbf{r}'\right)=\delta\left(\mathbf{r}-\mathbf{r}'\right)$$

$$\mathbf{r}'=\left(r_0\text{-i}b,\vartheta_0,\varphi_0\right)$$

$$G_s\left(\mathbf{r},\mathbf{r}'\right)\Big|_{\vartheta=\vartheta_B}=0 \text{ (acoustically soft; Dirichlet)}$$

$$\partial G_h\left(\mathbf{r},\mathbf{r}'\right)\Big/\partial\vartheta\Big|_{\vartheta=\vartheta_B}=0 \text{ (acoustically hard; Neumann)}$$

Complete eigenfunction expansion ansatz (scalar case):

$$G_\tau(\mathbf{r},\mathbf{r}')=\frac{2}{\pi}\int_0^\infty \lambda^2 \sum_{\sigma_\tau,m} a_{\sigma_\tau,m}\left(\lambda\right) j_{\sigma_\tau}(\lambda r)\, j_{\sigma_\tau}(\lambda r')Y_{\sigma_\tau,m}(\vartheta,\varphi)\, Y^*_{\sigma_\tau,m}(\vartheta',\varphi')d\lambda$$

Uniform CSB Diffraction by a Cone

$$G_\tau(\mathbf{r},\mathbf{r}') = \frac{2}{\pi}\int_0^\infty \lambda^2 \sum_{\sigma_\tau,m} a_{\sigma_\tau,m}(\lambda)\, j_{\sigma_\tau}(\lambda r)\, j_{\sigma_\tau}(\lambda r') Y_{\sigma_\tau,m}(\vartheta,\varphi)\, Y^*_{\sigma_\tau,m}(\vartheta',\varphi')d\lambda$$

Spherical Bessel functions of the 1st kind:

$$j_\sigma(x) = \sqrt{\pi/2x}\; J_{\sigma+1/2}(x)$$

Normalized surface spherical harmonics (circular cone):

$$Y_{\sigma_\tau,m}(\vartheta,\varphi) = I_{\sigma_\tau,m} P^m_{\sigma_\tau}(\cos\vartheta) e^{\mathrm{j}m\varphi}$$

Eigenvalues from conditions:

$$P^m_{\sigma_s}(\cos\vartheta)\Big|_{\vartheta=\vartheta_B} = 0 \text{ (acoustically soft)}; \quad \left.\partial P^m_{\sigma_h}(\cos\vartheta)\middle/\partial\vartheta\right|_{\vartheta=\vartheta_B} = 0 \text{ (acoustically hard)}$$

$$m = 0,\pm1,\pm2,\ldots;\ \sigma_\tau \in \mathbb{R}^+; \qquad m^2 < \sigma_\tau(\sigma_\tau+1)$$

Uniform CSB Diffraction by a Cone

Inserting

$$G_\tau(\mathbf{r},\mathbf{r}') = \frac{2}{\pi}\int_0^\infty \lambda^2 \sum_{\sigma_\tau,m} a_{\sigma_\tau,m}(\lambda)\, j_{\sigma_\tau}(\lambda r)\, j_{\sigma_\tau}(\lambda r')\, Y_{\sigma_\tau,m}(\vartheta,\varphi)\, Y^*_{\sigma_\tau,m}(\vartheta',\varphi')\, d\lambda$$

into

$$\Delta G_{\substack{s\\h}}(\mathbf{r},\mathbf{r}') + k^2 G_{\substack{s\\h}}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$$

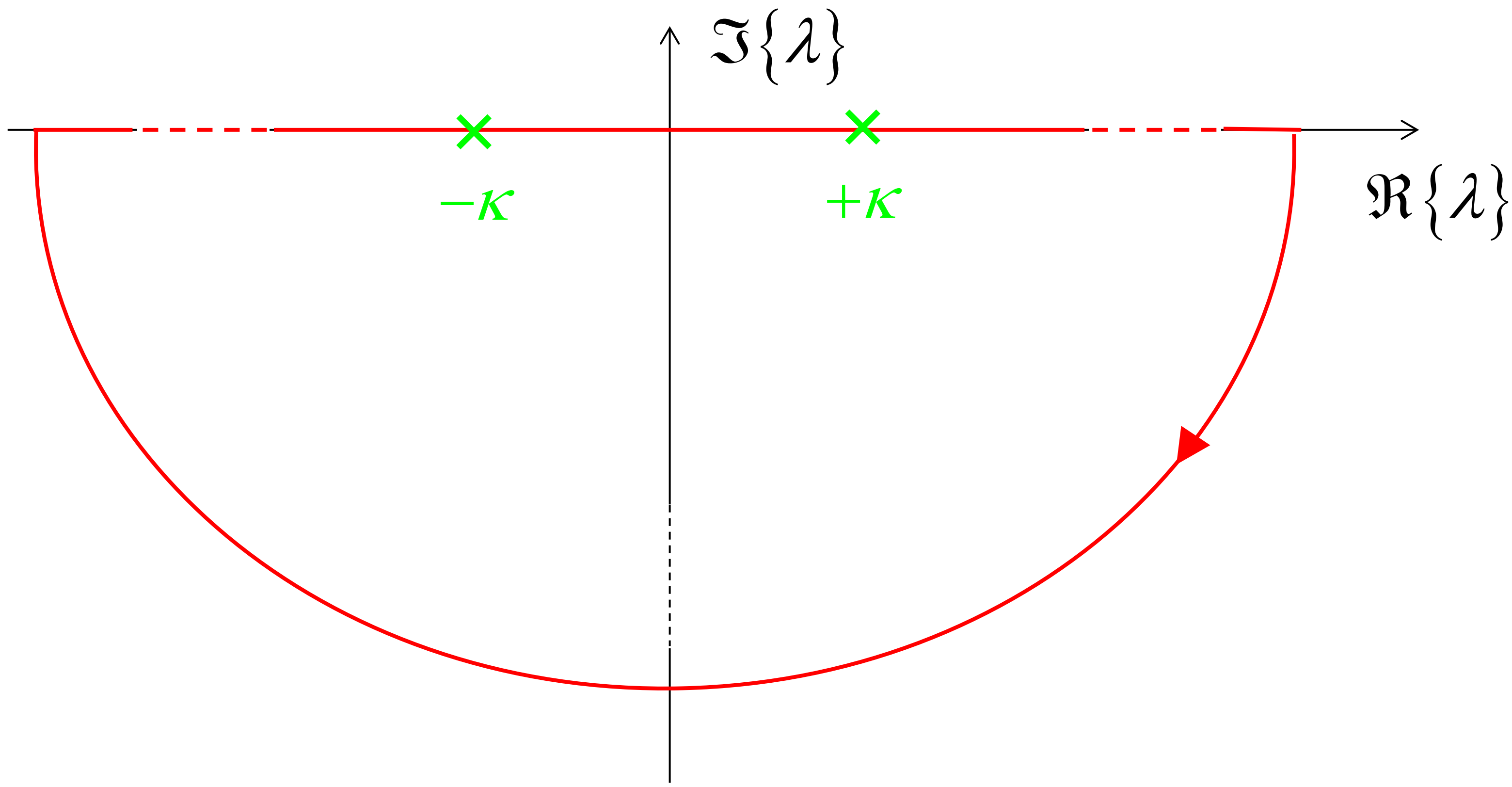
and using orthogonalities of the eigenfunctions yields the eigenfunction expansion:

$$G_\tau(\mathbf{r},\mathbf{r}') = \frac{2}{\pi}\int_0^\infty d\lambda \frac{\lambda^2}{\kappa^2-\lambda^2} \sum_{\sigma_\tau,m} j_{\sigma_\tau}(\lambda r)\, j_{\sigma_\tau}(\lambda r')\, Y_{\sigma_\tau,m}(\vartheta,\varphi)\, Y^*_{\sigma_\tau,m}(\vartheta',\varphi')$$

Uniform CSB Diffraction by a Cone

$$G_\tau(\mathbf{r},\mathbf{r}') = \frac{2}{\pi}\int_0^\infty d\lambda \frac{\lambda^2}{\kappa^2-\lambda^2}\sum_{\sigma_\tau,m} j_{\sigma_\tau}(\lambda r)\, j_{\sigma_\tau}(\lambda r')Y_{\sigma_\tau,m}(\vartheta,\varphi)\, Y^*_{\sigma_\tau,m}(\vartheta',\varphi')$$

Evaluation with residuum calculus in the complex λ-plane:



Bilinear symmetric form of the Green’s function of an acoustically hard or soft circular cone illuminated by a Uniform Complex-Source Beam:

$$G_\tau(\mathbf{r},\mathbf{r}') = 2\mathrm{i}\kappa \sum_{\sigma_\tau, m} j_{\sigma_\tau}(\kappa r)\, j_{\sigma_\tau}(\kappa r')\, Y_{\sigma_\tau, m}(\vartheta,\varphi)\, Y^*_{\sigma_\tau, m}(\vartheta',\varphi')$$

The corresponding Green’s function of the free space

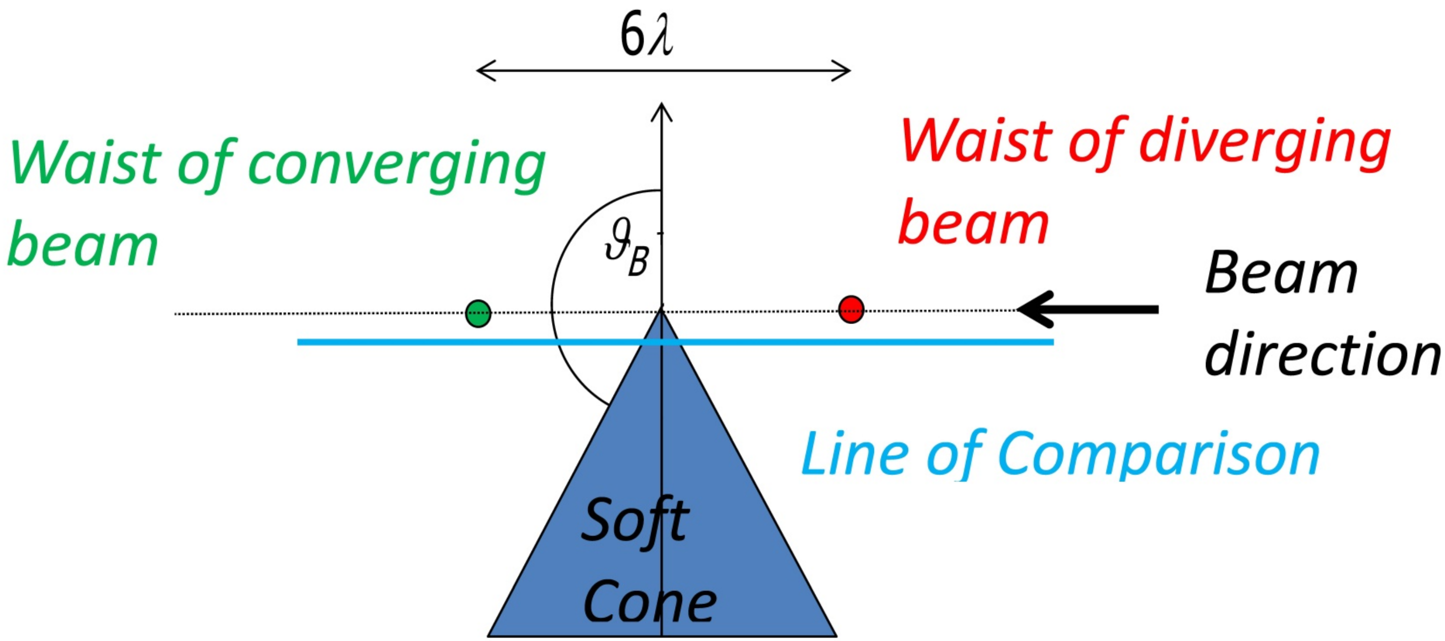
$$G(\mathbf{r},\mathbf{r}') = 2\mathrm{i}\kappa \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} j_n(\kappa r)\, j_n(\kappa r')\, Y_{n,m}(\vartheta,\varphi)\, Y^*_{n,m}(\vartheta',\varphi')$$

is identical to the heuristically obtained form:

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\left[\frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} + \frac{e^{-\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\right]$$

Validation:

Comparison of “Diverging” and “Converging” Incident CSB

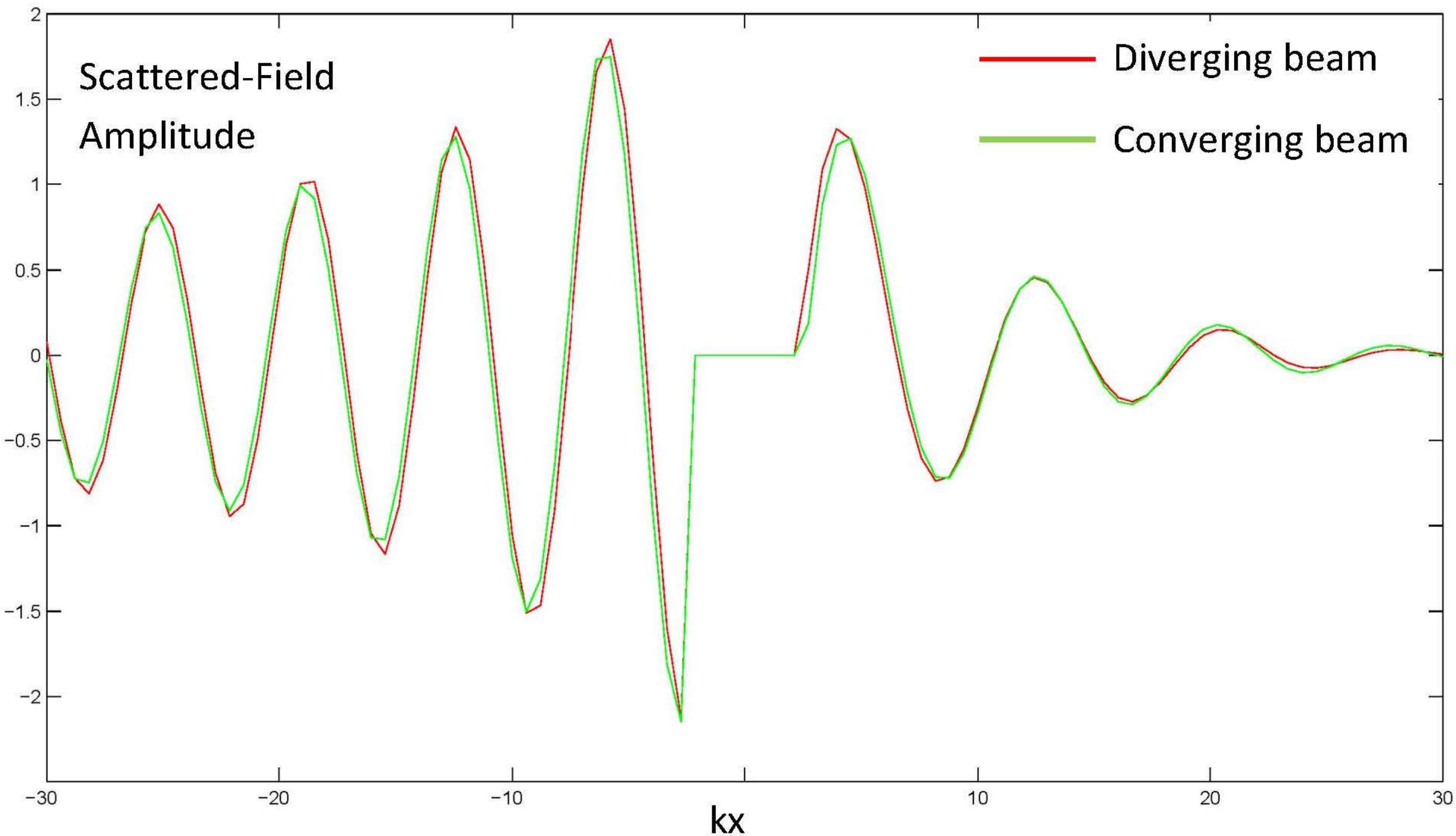


$kr_0 = 10, kb = 80, \vartheta_0 = 90°, \varphi_0 = 0°$ (diverging beam)

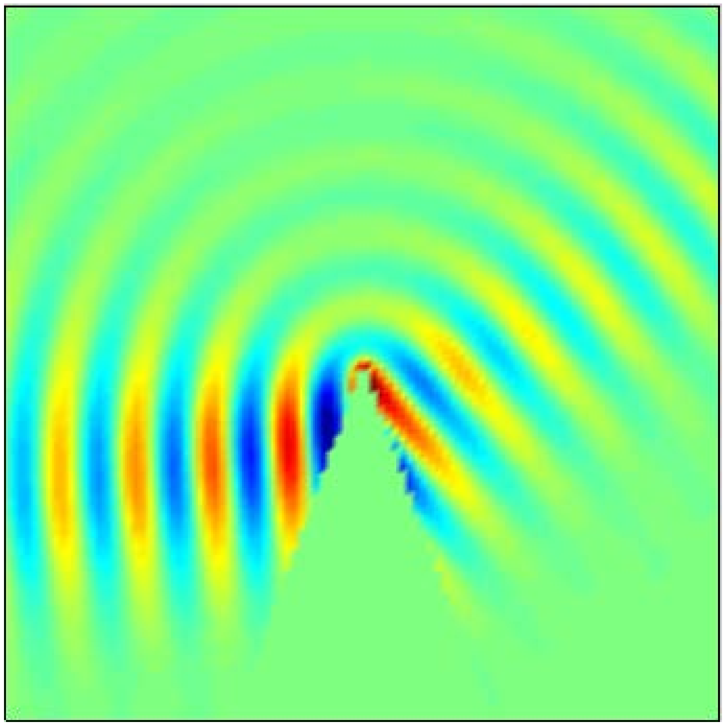
$kr_0 = -8.84955, kb = 80, \vartheta_0 = 90°, \varphi_0 = 0°$ (converging beam)

$\vartheta_B = 150°$

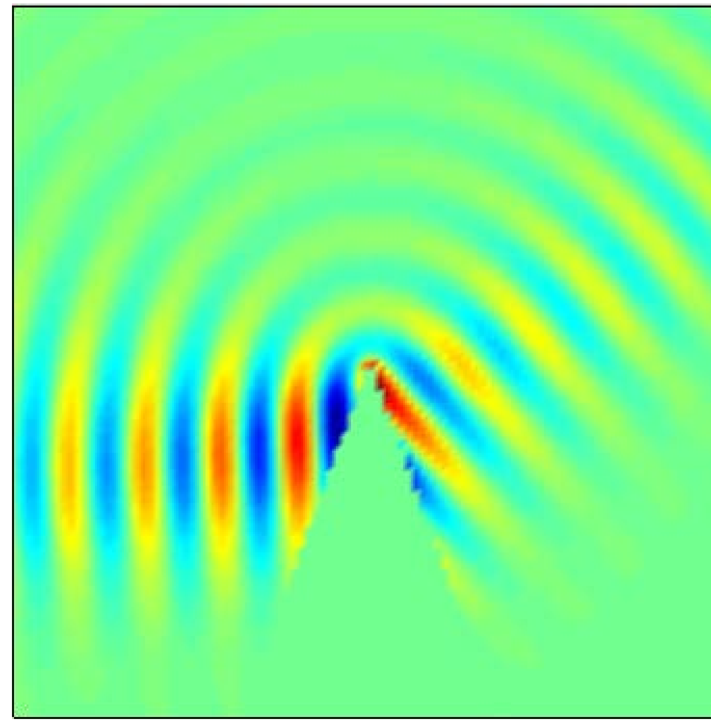
Validation: Comparison of the Scattered Fields for a "Diverging" and a "Converging" Incident CSB



Validation: Comparison of the Scattered Fields for a "Diverging" and a "Converging" Incident CSB

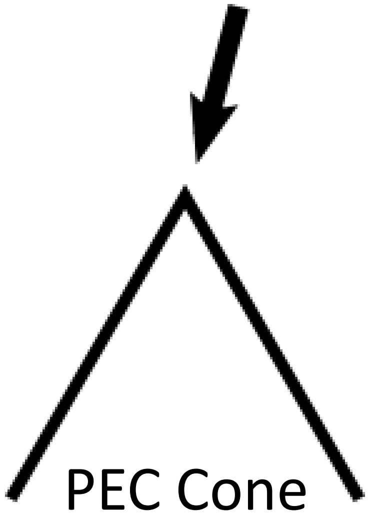


Diverging CSB



Converging CSB

Electromagnetic Uniform CSB Scattering by a PEC Cone:



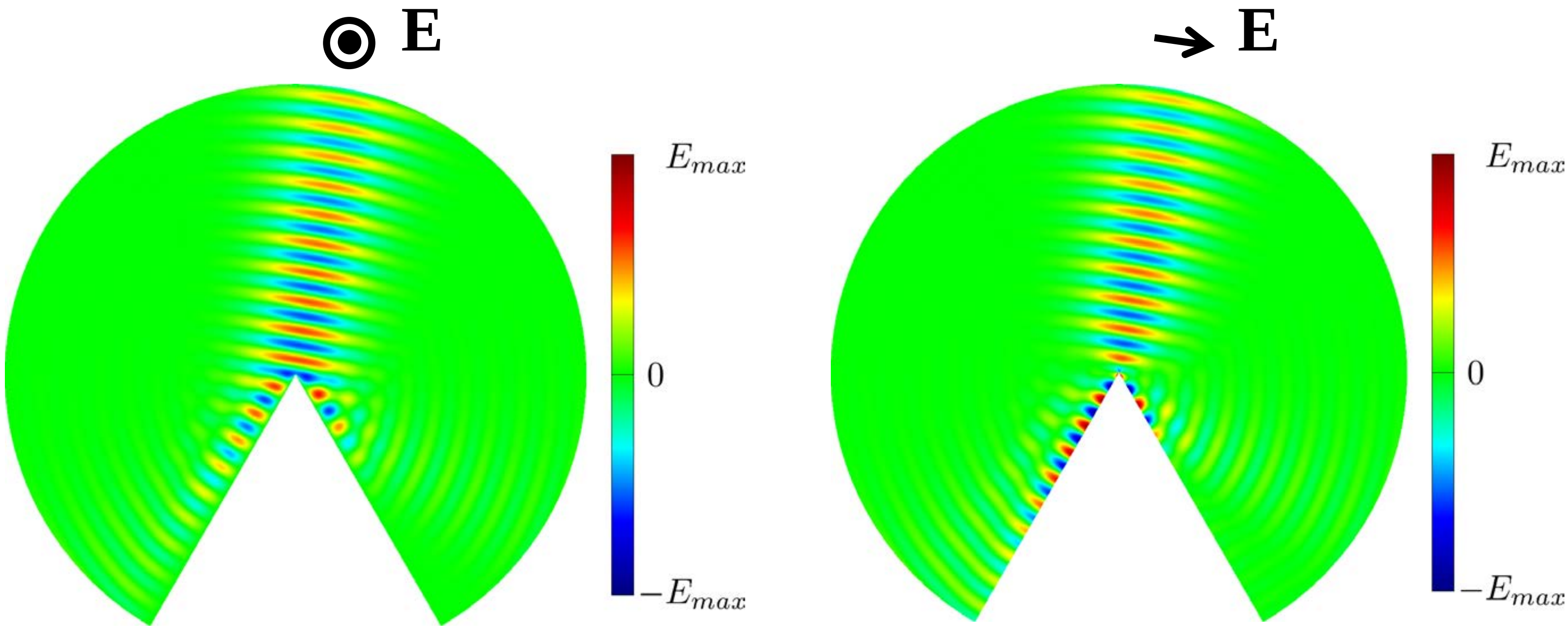
- Use dyadic Green's function of the PEC cone
- Complex Source:
  Hertzian Dipole at

$$r' = (0.001 - \mathrm{i}10)\Lambda$$

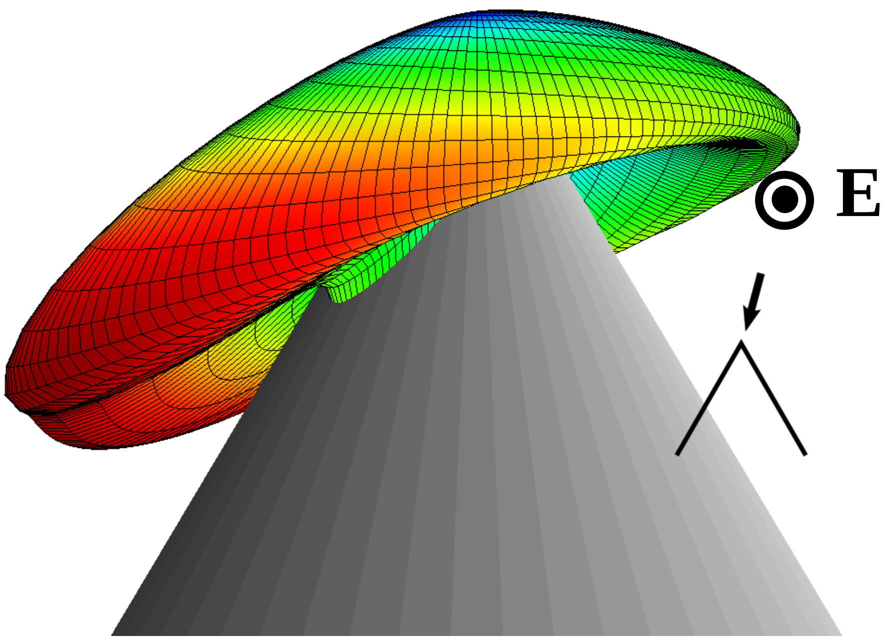
$$\vartheta' = 10^\circ$$

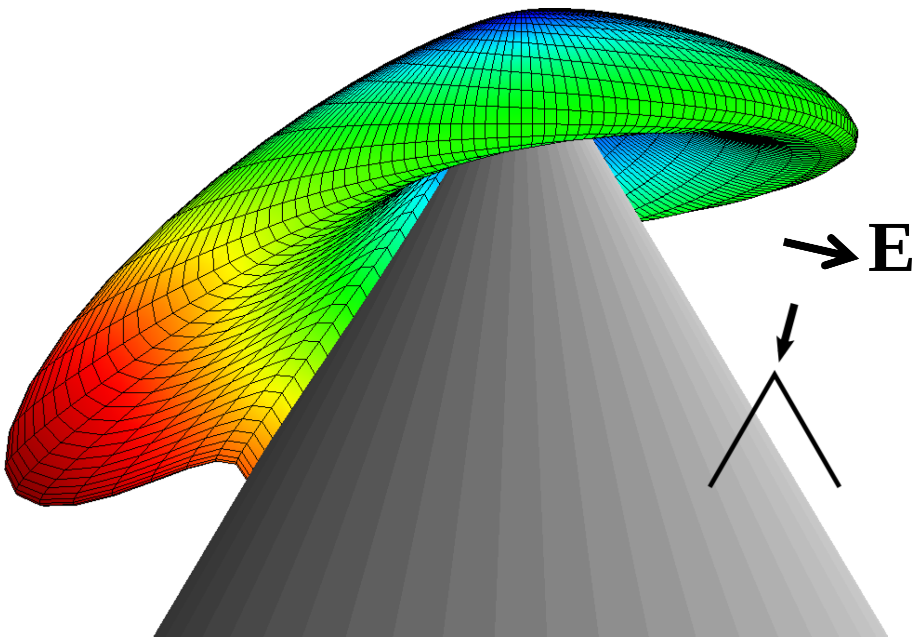
$$\varphi' = 0^\circ$$

Electromagnetic Uniform CSB Scattering of a by a PEC Cone:
Total Near Fields

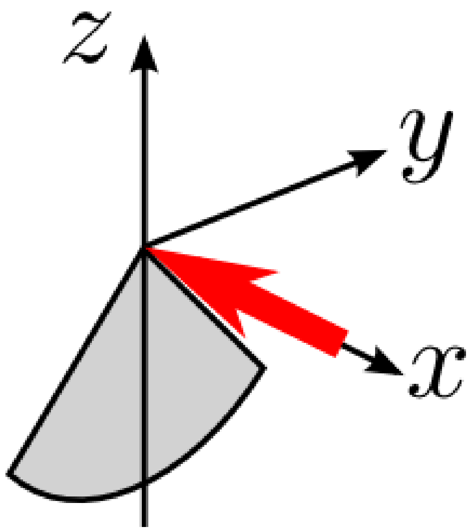


Electromagnetic Uniform CSB Scattering of a by a PEC Cone:
Scattered Far Fields





Electromagnetic Uniform CSB Scattering by a PEC Sector:



PEC Sector

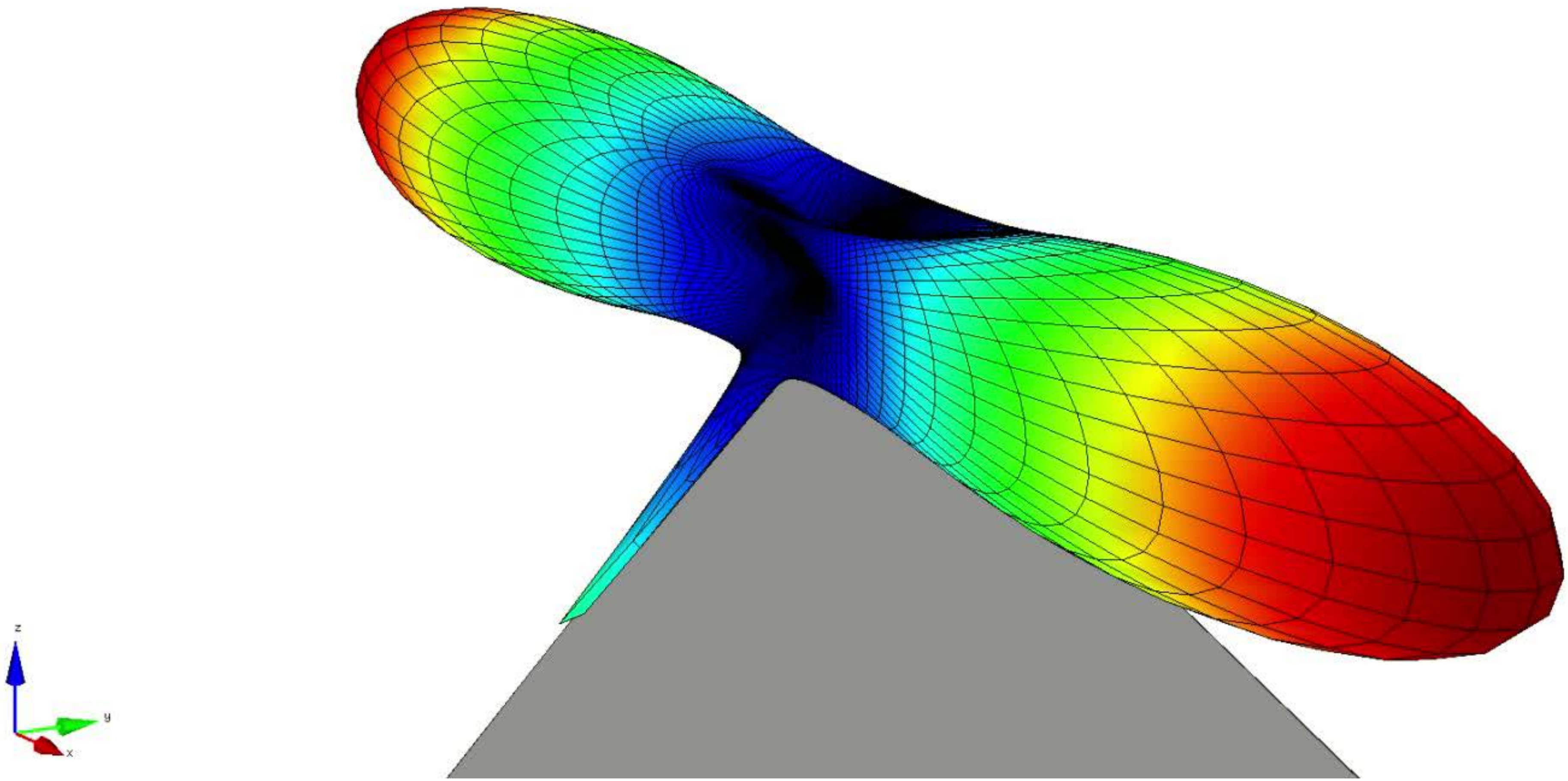
- Use dyadic Green's function of the PEC elliptic cone for $\vartheta_0 = \pi$ as a function of k^2

- Complex Source: Hertzian Dipole at

$$r' = (0.001 - \mathrm{i}10)\Lambda$$
$$\vartheta' = 90^\circ$$
$$\varphi' = 0^\circ$$

- Electric field polarized in *y*-direction.

Numerical Results



Conclusions

- Scattering of Uniform CSB by canonical objects (cone, wedge, etc.) can be described by a modified Green's function not satisfying the radiation condition,
- Uniform multipole representation (no case distinction) allows to probe an object at the beam's waist,
- Method allows to probe details (tip, edges, curved surfaces) by a localized plane wave,
- Extraction corresponding diffraction coefficients possible (e.g., comparison to results obtained by incident inhomogeneous plane waves).

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